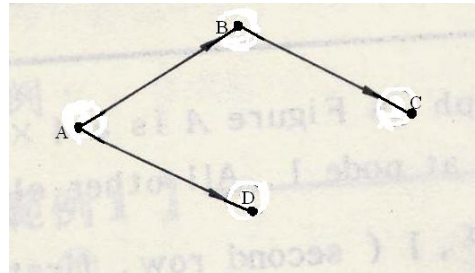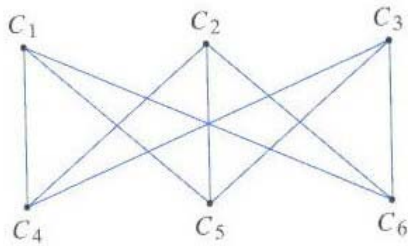# Chapter 1 Graph Theory

1-1 Representations of Graphs

Graph, $G=(V,E)$: It consists of the set V of vertices and the set E of edges. If each edge has its direction, the graph is called the directed graph (digraph). If each edge is undirected, the graph is called the undirected graph (multigraph).

Eg. An undirected graph (left) and a directed graph (right).

Path: A set of edges of which each edge connects to the initial vertex of the next edge is called a path.

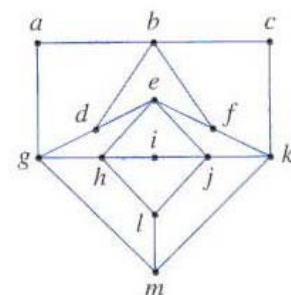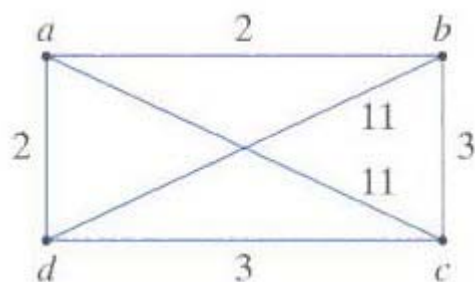**Length of a path:** The number of the edges in one path.
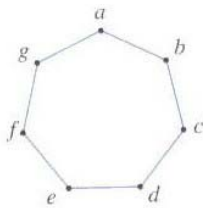
Eg. For the left graph, the path from a to d through b, c is 3, the path from g to b via a is 2.
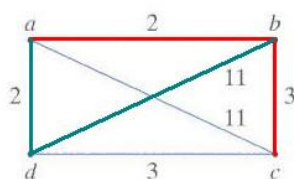
Weight, $w(v,x)$: The weighting value of edge from vertex v to vertex x.

Weighted graph: A graph's each edge has a weight.

Eg. A weighted graph (left) and an un-weighted graph (right).
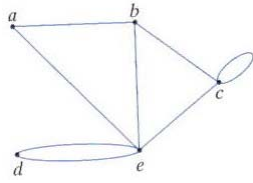

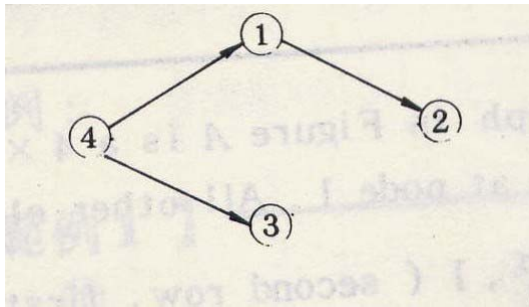**Weighted length of a path:** The sum of weights of the edges in one path.

Eg. For the left graph, the weighted length from a to c via b is 2+3=5. The weighted length from a to b via d is 2+11=13.

Adjacent matrix: $A=[A_{ij}]$, where A_{ij} is the number of paths from vertex i to vertex j.

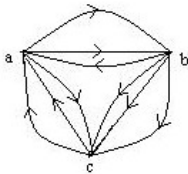**Eg. For the left graph, the corresponding adjacent matrix is**

$$
\begin{array}{c c}
 & \begin{array}{ccccc} a & b & c & d & e \end{array} \\
\begin{array}{c} a \\ b \\ c \\ d \\ e \end{array} &
\left(\begin{array}{ccccc}
0 & 1 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 \\
0 & 1 & 2 & 0 & 1 \\
0 & 0 & 0 & 0 & 2 \\
1 & 1 & 1 & 2 & 0
\end{array}\right)
\end{array}
$$

Eg. For the left digraph, the corresponding adjacent matrix is

$$
\begin{array}{c c}
 & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} &
\left[\begin{array}{cccc}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0
\end{array}\right]
\end{array}
$$

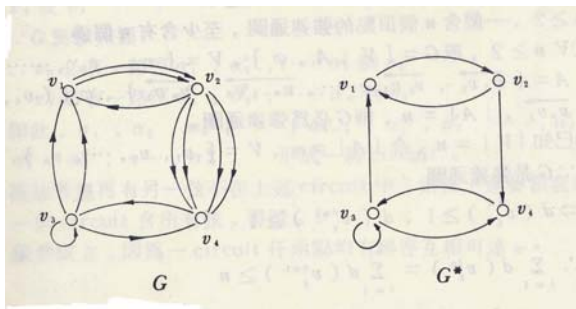**Multiplicity:** The number of the edges from vertex v to vertex u.

Eg.   The adjacent matrix of the left graph is
$$
\begin{array}{c c}
 & \begin{array}{ccc} a & b & c \end{array} \\
\begin{array}{c} a \\ b \\ c \end{array} &
\left[\begin{array}{ccc}
0 & 2 & 1 \\
1 & 0 & 3 \\
2 & 0 & 0
\end{array}\right]
\end{array}
$$ **. The**

multiplicity of (a,b) is 2 and the multiplicity of (b,c) is 3.

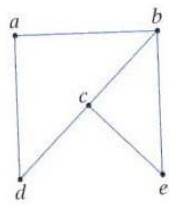**Simple graph:** The graph whose entry of the adjacent matrix is 0 or 1 is called a simple graph.

Eg. The graph G^* is a simple graph, but G is not a simple graph. The corresponding adjacent matrix of G

is $M=$
$$
\left[\begin{array}{cccc}
0 & 2 & 0 & 0 \\
1 & 0 & 0 & 4 \\
2 & 0 & 1 & 1 \\
0 & 0 & 1 & 0
\end{array}\right]
$$ ** and the**

corresponding adjacent matrix of G^* is $M^*=$
$$
\left[\begin{array}{cccc}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 \\
1 & 0 & 1 & 1 \\
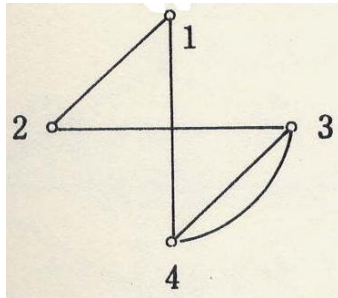0 & 0 & 1 & 0
\end{array}\right]
$$ **.**

Theorem If A is the adjacent matrix of a simple graph, the ij_{th} entry of A^n is equal to the number of paths of length n from v_i to v_j. And the diagonal entries of A^2 give the number of the edges on the vertices.

Eg. For the left simple graph, the adjacent matrix A is

$$A = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{ccccc} a & b & c & d & e \\ \left(0 \right. & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ \left. 0 \right. & 1 & 1 & 0 & 0 \end{array} \!\!\right)$$

$$A^2 = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{ccccc} a & b & c & d & e \\ \left(2 \right. & 0 & 2 & 0 & 1 \\ 0 & 3 & 1 & 2 & 1 \\ 2 & 1 & 3 & 0 & 1 \\ 0 & 2 & 0 & 2 & 1 \\ \left. 1 \right. & 1 & 1 & 1 & 2 \end{array} \!\!\right)$$

$A_{13}=2$ means that there are **2 paths** of **length 2** from a to c: $a{\to}b{\to}c$ and $a{\to}d{\to}c$. The diagonal entries of A^2 give the number of the edges on the vertices. For example, $[A^2]_{33}=3$ means that there are **3 edges** incident on c.

Eg. For the left graph, the adjacent matrix A is

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 \\ 1 & 0 & 2 & 0 \end{bmatrix} \quad \text{and} \quad A^2 = \begin{bmatrix} 3 & 1 & 3 & 1 \\ 1 & 2 & 0 & 3 \\ 3 & 0 & 5 & 0 \\ 1 & 3 & 0 & 5 \end{bmatrix}. \quad A^2{}_{13}=3$$
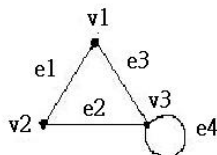
means that there are **3 paths** of **length 2** from vertex 1 to vertex 3: $1{\to}2{\to}3$, $1{\to}4{\to}3$ (straight line), and $1{\to}4{\to}3$ (curve).

Note: This graph is not a simple graph, so that there are **3 edges** incident on vertices 3, 4 but $[A^2]_{33}=[A^2]_{44}=5$.
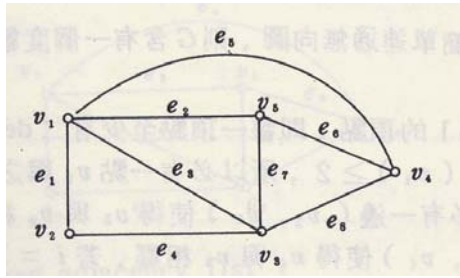
Incident matrix: A matrix M representing a graph of which row is labeled with the vertices and column is labeled with the edges. $M=[M_{ij}]$ where

$$M_{ij} = \begin{cases} 1, & \text{if } e_j \text{ is incident on } v_i \\ 0, & \text{else} \end{cases}$$

Eg. The incident matrix of the left graph is

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \end{array} \begin{array}{cccc} e_1 & e_2 & e_3 & e_4 \\ \left[1 \right. & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ \left. 0 \right. & 1 & 1 & 1 \end{array} \!\!\right].$$

Eg. For the left graph, (a) find the adjacent matrix A, and (b) find the incident matrix M.

(Sol.) (a) $A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix}$ (b) $M = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$
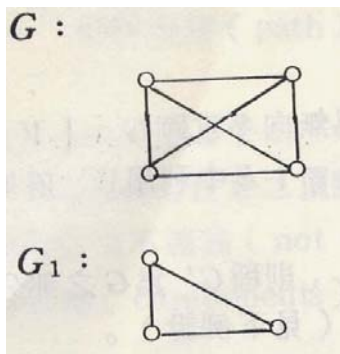
Subgraph: $G=(V',E')$ is a subgraph of $G=(V,E)$ if (a) $V' \subseteq V$ and $E' \subseteq E$. (b) $\forall\, e' \in E'$, if e' is incident on v' and w', then v', $w' \in V'$.

Eg. In the following figure, G_1 is a subgraph of G. G_2 is a partial graph of G but it is not a subgraph of G because $\overline{de}$ does not exist.
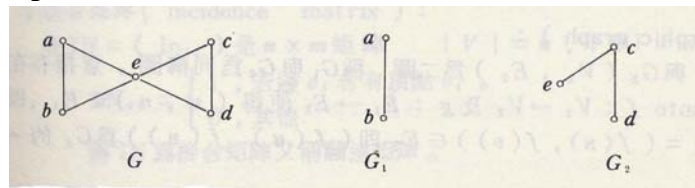
Component: The subgraph G' of G consisting of all edges and vertices in G that are contained in some path beginning at v is called the component of G containing v.
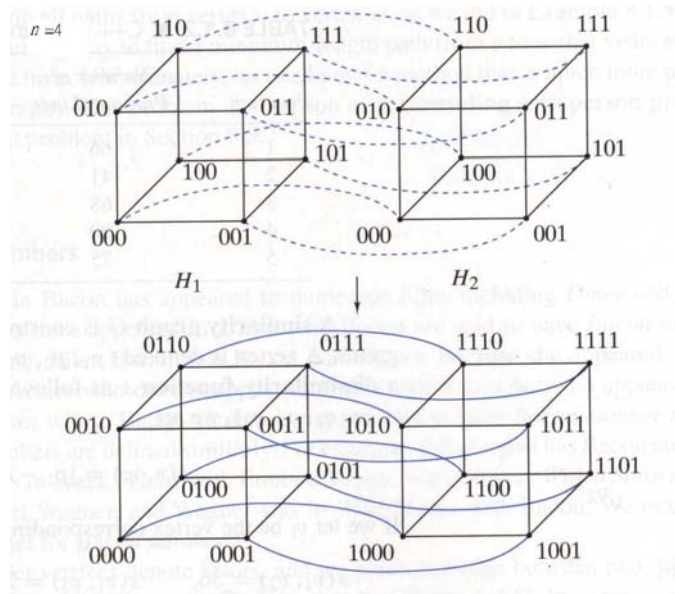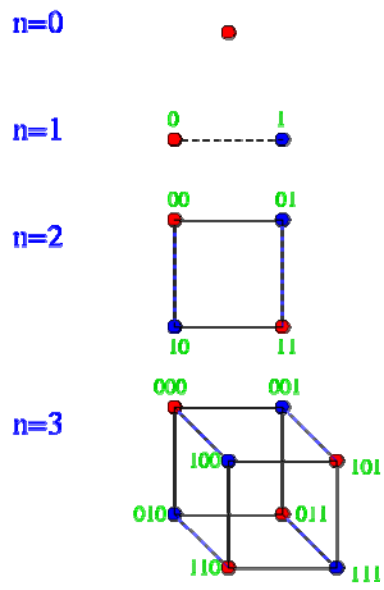
Complement: $G_1=(V_1,E_1)$ is a subgraph of $G=(V,E)$. $G_2=(V_2,E_2)$ is the complement of G_1 if $E_2=E- E_1$ and V_2 contains only the vertices of the edges in E_2.

G :

G_1 :

Eg. For the left G and G_1, what is the complement of G_1?

(Sol.)

Hypercube (*n*-cube):

n=0

n=1

0   1

n=2

00   01

10   11

n=3

000   001

100   101

010   011

110   111

$n = 4$

110   111   110   111

010   011   010   011

100   101   100   101

000   001   000   001

H_1 H_2

0110   0111   1110   1111

0010   0011   1010   1011

0100   0101   1100   1101

0000   0001   1000   1001
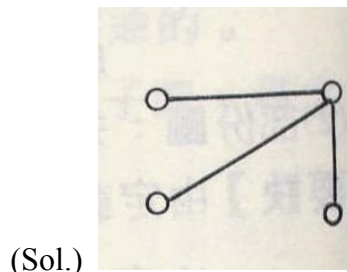
1-2 Connected Graphs and Cycles

Connected graph: Any vertices v and w in the graph, there exists at least one path from v to w.

Strongly connected graph: Any vertices v and w in the digraph, there exists at least one path from v to w and there exists at least one another path from w to v.
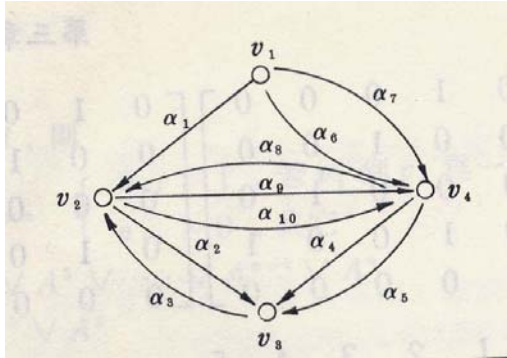
Eg. (a) Find the corresponding simple graph of G.

(b) Is it strongly connected?

(Sol.) (a)
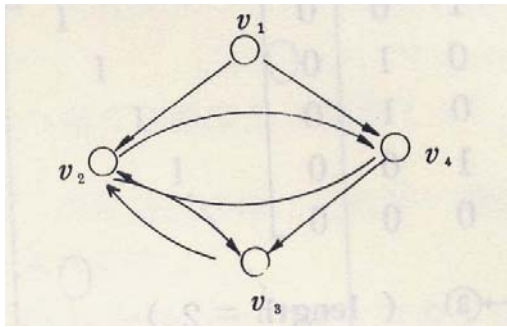
(b) No, $\because$ v_1 can achieve the other vertices but the other vertices can not achieved v_1, $\therefore$ it is not strongly connected.

Eg. Which is the strongly connected graph in the following figures?
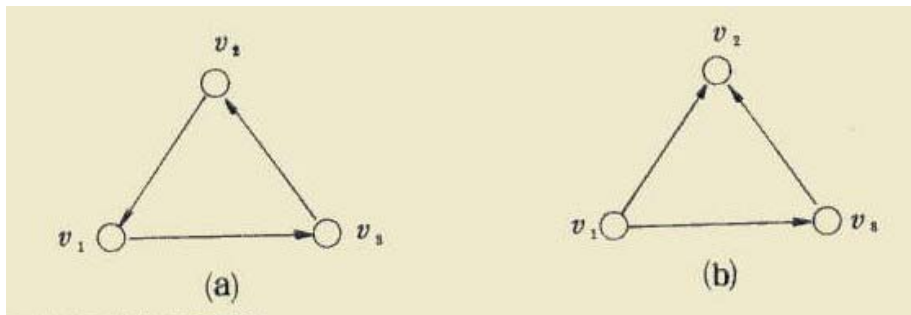
(Sol.) $\because$ v_2 can achieve v_1 directly and v_1 can achieve v_2 via v_3, $\therefore$ Figure (a) is strongly connected.

$\because$ v_2 can not achieve v_1 and v_3, $\therefore$ Figure (b) is not strongly connected..

Theorem Let G be a graph without isolated point, then G is strongly connected if and only if there exists a circuit comprising all directed edges at least one time.

Theorem $\forall\, n \geq 2$, there is a strongly connected graph that has n edges.

Theorem $\forall\, n \geq 2$, an *n*-vertex strongly connected graph has at least *n* edges.

Degree, δ(*v*) (or *deg(v)*, or *d*ᵥ): The number of the edges incident on the vertex.

Indegree, d_v^{in}: The number of the edges entering a certain vertex.

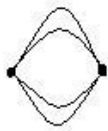**Outdegree, d_v^{out}:** The numbers of the edges flowing out from a certain vertex.

Eg. In the following figure, *deg(A)*=2, *deg(B)*=3, *deg(C)*=3, *deg(D)*=3 in Graph (a) and *deg(D)*=2 in Graph (b).

(a)   (b)
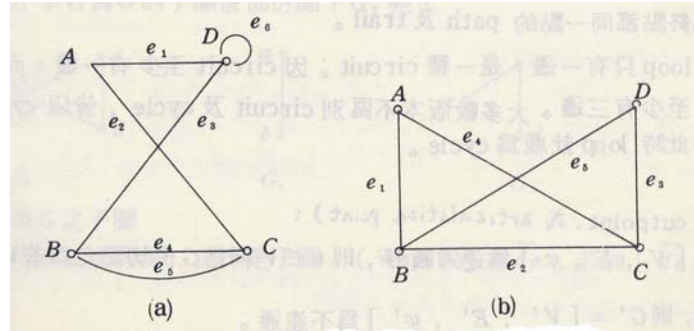
Theorem **For an undirected graph *G[V,E]*, where |*V*|=*n* and |*E*|=*e*. We have**

$$\sum_{i=1}^{n}\delta(v_i)=2e.$$

Eg. For the left directed graph, |*V*|=2 and |*E*|=4. We have

$$\sum_{i=1}^{n}\delta(v_i)=4+4=8$$

Theorem **For a directed graph *G[V,E]*, where |*V*|=*n* and |*E*|=*m*.**

$$\sum_{i=1}^{n}d_{v_i}^{\,in}=\sum_{i=1}^{n}d_{v_i}^{\,out}=m.$$
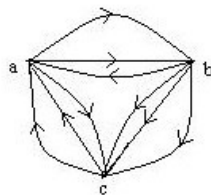
Eg. For the left directed graph, |*V*|=3 and |*E*|=9. We have

$$\sum_{i=1}^{n}d_{v_i}^{\,in}=3+2+4=9=\sum_{i=1}^{n}d_{v_i}^{\,out}=3+4+2$$

Simple path: A path has no repeatedly-visited edge.

Elementary path: A path has no repeatedly-visited vertex.

Cycle (or Circuit): A finite path whose initial vertex and terminal are the same is called a cycle or circuit.

Simple circuit: A circuit has no repeated edge.

Simple cycle: A circuit has no repeated vertex.

Loop: A cycle of length=1.
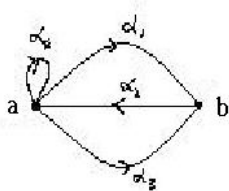
Eg. A path: $a\alpha_4 a$ **is a loop.**

Eg. A path: $a\alpha_1 b\alpha_2 a$ **is a simple circuit and a simple cycle. Another path:** $b\alpha_2 a\alpha_1 b$ **is also a simple circuit and a simple cycle.**

**Eg. A path:** $a\alpha_4 a\alpha_1 b\alpha_2 a$ **is a simple circuit but not a simple cycle.**

Hamiltonian cycle (Traveling salesman problem): A cycle in Graph G that contains **each vertex** in G **exactly once**, except for the starting and ending vertex that appears twice.

Theorem An undirected simple graph with n vertices ($n \geq 3$) is Hamiltonian if every vertex has degree $n / 2$ or greater.

Eg. Two examples of graphs with Hamiltonian cycles.

 : $a\to b\to c\to d\to e\to f\to g\to a$

 : $a\to b\to c\to d\to e\to a$
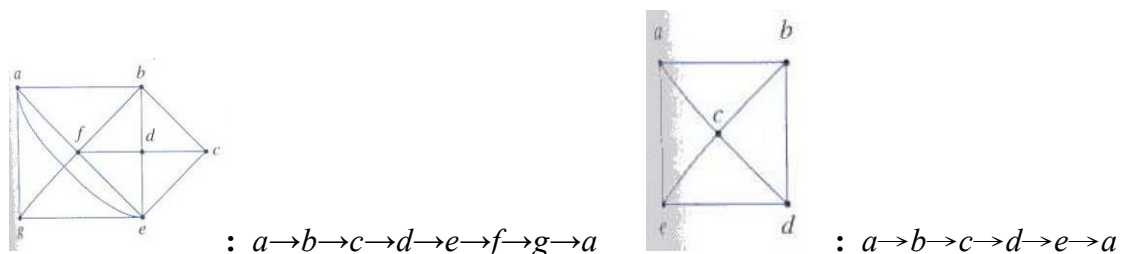
Eg. Four examples of graphs with no Hamiltonian cycles.
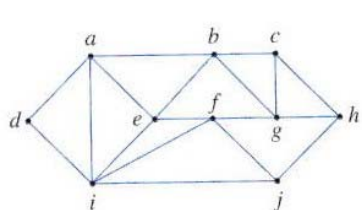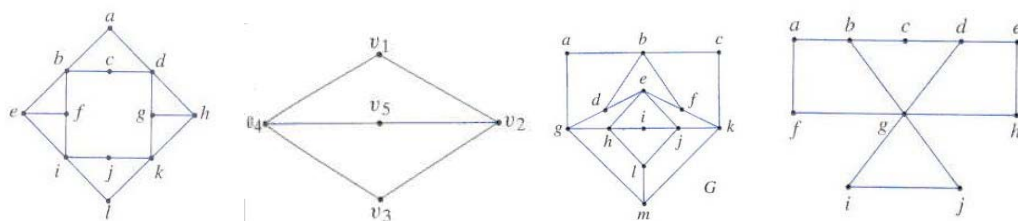
Eg. Find a Hamiltonian cycle in the left graph.

(Sol.) $d\to a\to e\to b\to c\to h\to g\to f\to j\to i\to d$
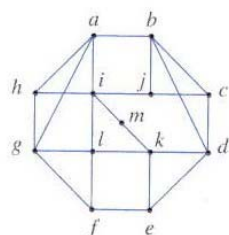
Eg. Find a Hamiltonian cycle in the left graph.

(Sol.) $a\to b\to c\to j\to i\to m\to k\to d\to e\to f\to l\to g\to h\to a$
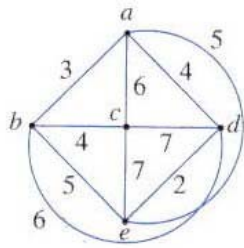
Eg. Find a Hamiltonian cycle in the left graph.

(Sol.) $e{\rightarrow}d{\rightarrow}a{\rightarrow}b{\rightarrow}c{\rightarrow}e$
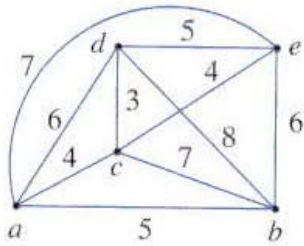
Eg. Solve the traveling salesman problem (Find a Hamiltonian cycle) for the left graph.

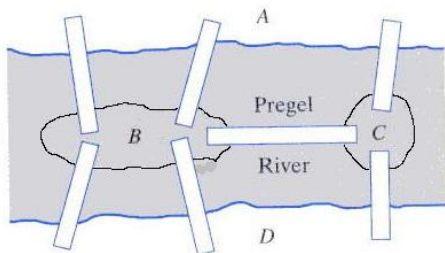(Sol.) $e{\rightarrow}b{\rightarrow}a{\rightarrow}c{\rightarrow}d{\rightarrow}e$
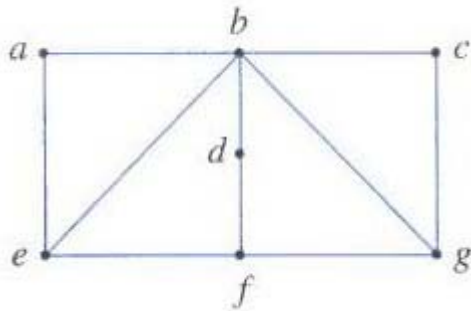
Eg. Show that the graph has no Hamiltonian cycle.

(Proof) If b is not initial or terminal vertex, b must be passed twice at least. If b is initial or terminal vertex, b must be passed three times at least.
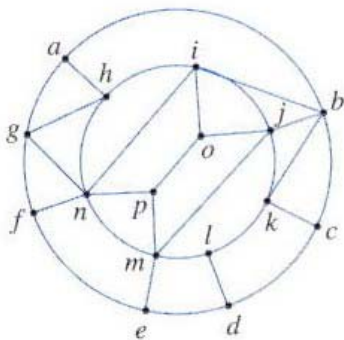
Eg. Find a Hamiltonian cycle in the left graph.

(Sol.) $a{\rightarrow}b{\rightarrow}c{\rightarrow}d{\rightarrow}e{\rightarrow}f{\rightarrow}n{\rightarrow}p{\rightarrow}m{\rightarrow}l{\rightarrow}k{\rightarrow}j{\rightarrow}o$ $\rightarrow i{\rightarrow}h{\rightarrow}g{\rightarrow}a$

The bridge of Konigsberg:

In 1736, Euler presented a problem: There are 2 islands and 7 bridges as shown in figure. How does a salesman pass each bridge once if he wanted to visit 4 customers lived in A, B, C, and D.
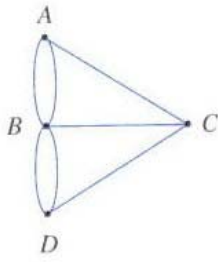
The problem can be simplified into a graph model as shown in the left graph. And the question is: "Can we draw it without interruption?" The answer is "No". That is, there is no Euler cycle in this graph. But the graph has a Hamiltonian cycle. A salesman can visit 4 customers as the following path: $A \to B \to D \to C \to A$.

Euler trail: A simple path passing each edge once in the graph.

Euler cycle (or Euler circuit): A simple path, with the same initial vertex and terminal, passes **each edge** in G **exactly once**. And we can draw an Euler cycle without interruption.

Theorem **For a directed graph without isolated point, it has an Euler cycle if and only if the graph is connected and $d_v^{in} = d_v^{out}$.**
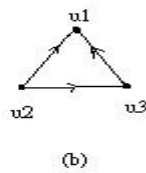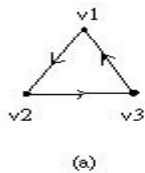
Eg. Graph (a) has an Euler cycle because $d_v^{in} = d_v^{out} = 1$ for each vertex. But Graph (b) has an no Euler cycle because $d_v^{in} = 2 \neq 0 = d_v^{out}$ for vertex u_1 and $d_v^{in} = 0 \neq 2 = d_v^{out}$ for vertex u_2.

Theorem **If an undirected graph has an Euler cycle, then it is connected and every vertex has even degree.**

Eg. Left Graph (a) has an Euler cycle because the degree is 4 for each vertex. But Graph (b) has not an Euler cycle because the degree is 3 for each vertex.
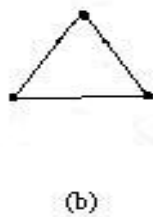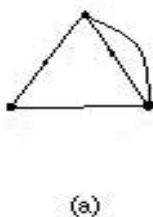
Eg. Left Graph (a) has not an Euler cycle because the degrees of two vertices are 3. Graph (b) has an Euler cycle because the degree is 2 for each vertex.
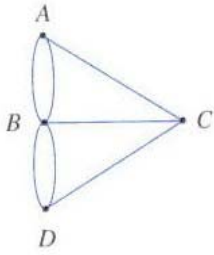
Eg. Show that no solution for the problem: The bridge of Konigsberg.

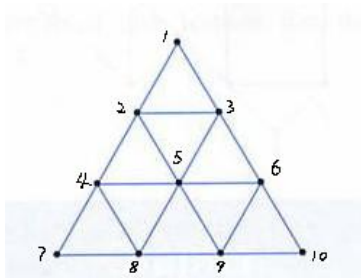(Proof) The left graph shows that the degree of each vertex is odd. There is no Euler cycle.
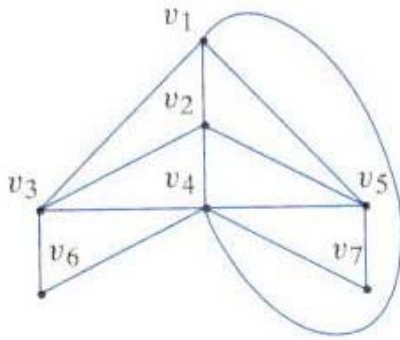
Eg. Find an Euler cycle in the left graph.

(Sol.) $v_6{\rightarrow}v_4{\rightarrow}v_7{\rightarrow}v_5{\rightarrow}v_1{\rightarrow}v_3{\rightarrow}v_4$

${\rightarrow}v_1{\rightarrow}v_2{\rightarrow}v_5{\rightarrow}v_4{\rightarrow}v_2{\rightarrow}v_3{\rightarrow}v_6$

Or, $v_1{\rightarrow}v_2{\rightarrow}v_5{\rightarrow}v_1{\rightarrow}v_3{\rightarrow}v_2{\rightarrow}v_4$

${\rightarrow}v_5{\rightarrow}v_7{\rightarrow}v_4{\rightarrow}v_3{\rightarrow}v_6{\rightarrow}v_4{\rightarrow}v_1$

Eg. Find an Euler cycle in the left graph.

(Sol.)

$10{\rightarrow}9{\rightarrow}6{\rightarrow}5{\rightarrow}9{\rightarrow}8{\rightarrow}5{\rightarrow}4{\rightarrow}8{\rightarrow}7$

${\rightarrow}4{\rightarrow}2{\rightarrow}5{\rightarrow}3{\rightarrow}2{\rightarrow}1{\rightarrow}3{\rightarrow}6{\rightarrow}10$
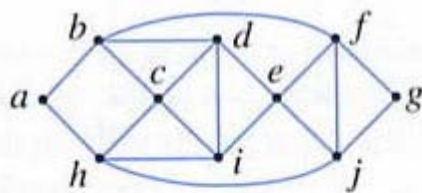
Eg. Find the Euler cycles in the left graphs.

(Sol.) (a) $a{\rightarrow}b{\rightarrow}d{\rightarrow}c{\rightarrow}b{\rightarrow}f{\rightarrow}g{\rightarrow}j{\rightarrow}f$

${\rightarrow}e{\rightarrow}j{\rightarrow}h{\rightarrow}c{\rightarrow}i{\rightarrow}d{\rightarrow}e{\rightarrow}i{\rightarrow}h{\rightarrow}a$

(b) $a{\rightarrow}b{\rightarrow}c{\rightarrow}b{\rightarrow}d{\rightarrow}e{\rightarrow}h{\rightarrow}f{\rightarrow}i{\rightarrow}j$

${\rightarrow}k{\rightarrow}i{\rightarrow}h{\rightarrow}g{\rightarrow}f{\rightarrow}e{\rightarrow}g{\rightarrow}d{\rightarrow}c{\rightarrow}a$
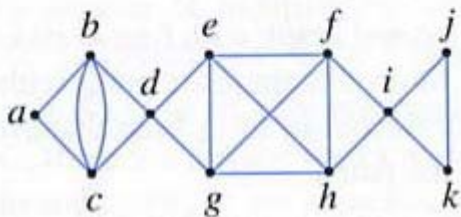
$$\begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Eg. Consider the adjacent matrix. (Assume that *a, b, c, d, e,* and *f* represent the vertices). (a) Is the graph connected? (b) Is the graph strongly connected? (c) Does the graph have an Euler cycle? (d) Is there are 3 paths from *a* to *f*? [台科大資工所]

1-3 Isomorphism of Graphs

Isomorphic graphs: G_1 and G_2 are isomorphic if there is a one-to-one, onto function f from the vertices of G_1 to the vertices of G_2, and a one-to-one, onto function g from the edges of G_1 to the edges of G_2, so that an edge e is incident on v and w in G_1 if and only if the edge $g(e)$ is incident on $f(v)$ and $f(w)$.

Theorem G_1 and G_2 are isomorphic if they have the same incident matrix.

Eg. The following graphs G_1 and G_2 are isomorphic.
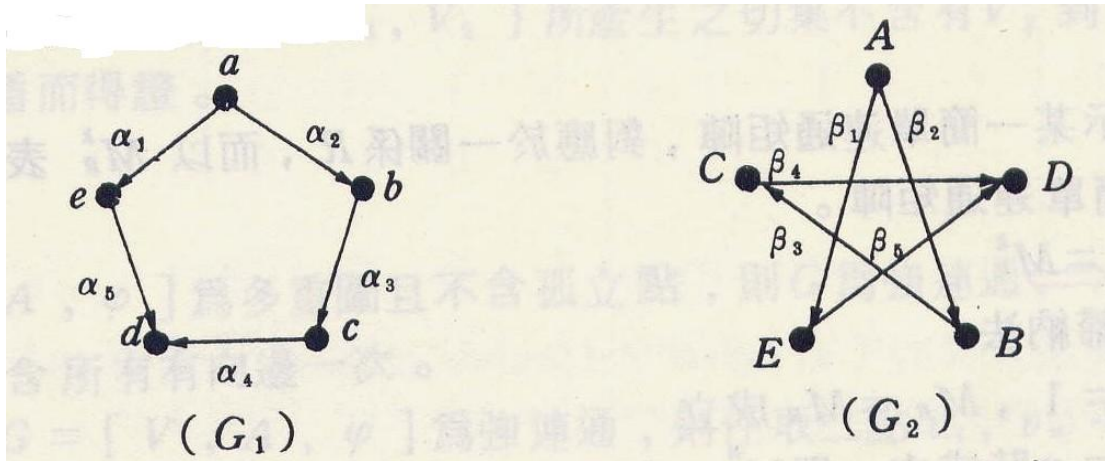
And G_1 and G_2 have the same incident matrix $M = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$

Eg. Determine whether the graphs (1) and (2) are isomorphic?
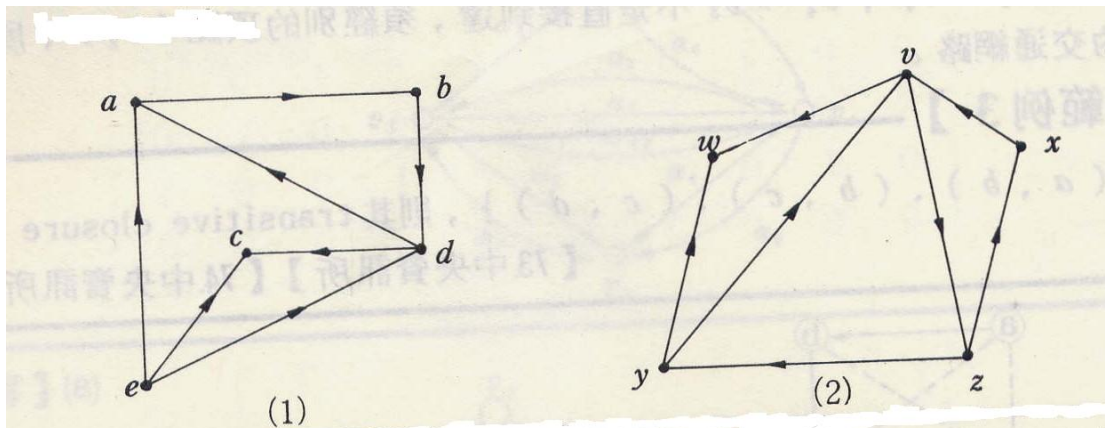
(Sol.) No. $\because d_v{}^{in}(a)=2$ and $d_v{}^{out}(a)=1$ in Graph (1), but $d_v{}^{in}(v)=2$ and $d_v{}^{out}(v)=2$, $d_v{}^{in}(w)=1$ and $d_v{}^{out}(w)=1$, $d_v{}^{in}(x)=1$ and $d_v{}^{out}(x)=1$, $d_v{}^{in}(y)=1$ and $d_v{}^{out}(y)=2$, $d_v{}^{in}(z)=1$ and $d_v{}^{out}(z)=2$ in Graph (2), $\therefore$ Graphs (1) and (2) are not isomorphic.

Eg. Determine whether the graphs G_1 and G_2 are isomorphic?
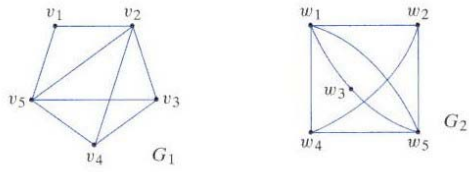
: yes, $(v_1,v_2,v_3,v_4,v_5) \rightarrow (w_3,w_1,w_4,w_2,w_5)$
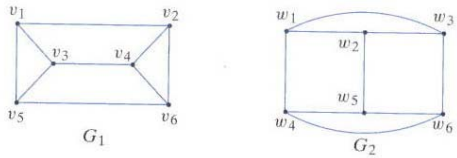
: yes, $(v_1,v_2,v_3,v_4,v_5,v_6) \rightarrow (w_3,w_6,w_2,w_5,w_1,w_4)$

Eg. Determine whether the graphs G_1 and G_2 are isomorphic?
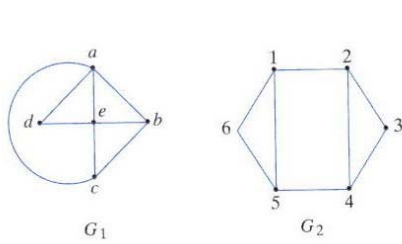
: no
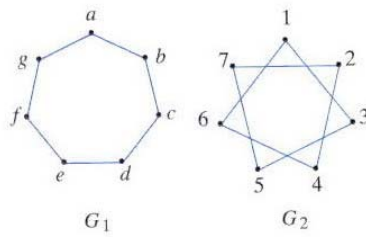
: yes
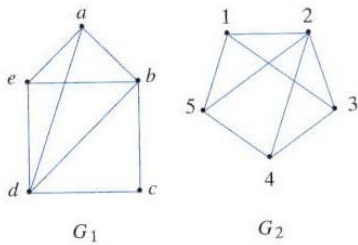
: no
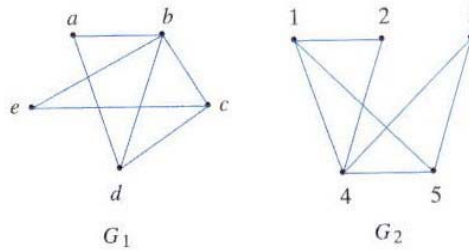
: yes
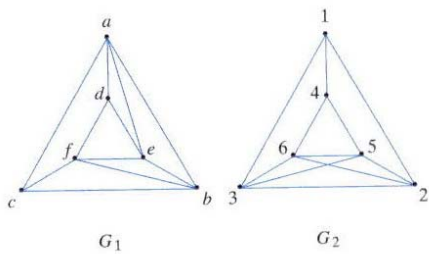
: no
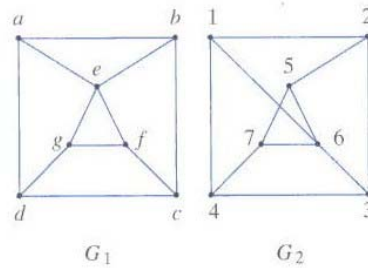
: no

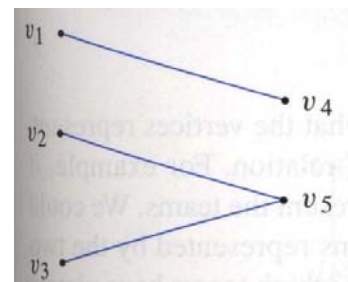**1-4 Planar Graphs**

Planar graph: A Graph G is planar if G can be drawn in the plane without its edges crossing

Eg. Two examples of planar graphs.
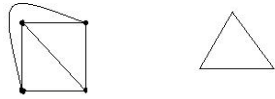
Bipartite graph: $G[V,E]$ is bipartite if $\exists V_1$ and V_2 such that $V_1 \cap V_2 = \phi$, $V_1 \cup V_2 = V$, and each edge in E is incident on one vertex in V_1 and one vertex in V_2.

Eg. Two examples of bipartite graphs.

Complete bipartite graph on m and n vertices, $K_{m,n}$: A simple graph whose vertex set is partitioned into set V_1 with m vertices and V_2 with n vertices in which the edge set consists of all edges of the form (v_1, v_2) with $v_1 \in V_1$ and $v_2 \in V_2$.

Eg. An example of complete bipartite graph $K_{2,4}$
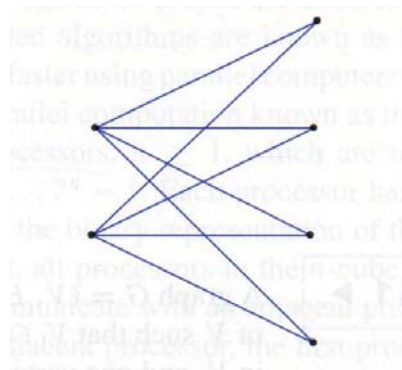
Complete graph, K_n: The simple graph with n vertices in which there is an edge between every pair of distinct vertices.

Eg. The complete graph K_4 with 4 vertices.

Eg. Two examples of nonplanar graphs K_5 and $K_{3,3}$.

: K_5

: $K_{3,3}$
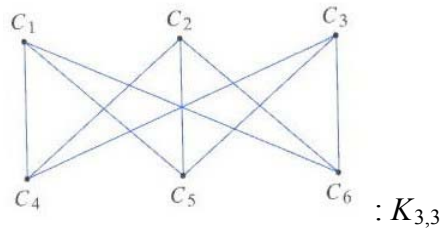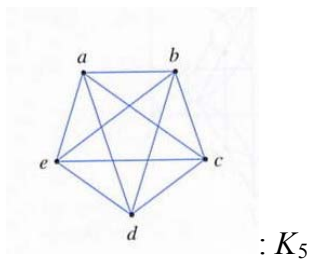
Kuratowski's Theorem A graph G is planar if and only if G does not contain a subgraph homeomorphic to K_5 or $K_{3,3}$.

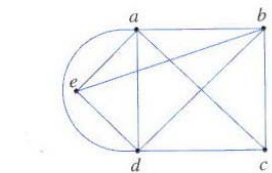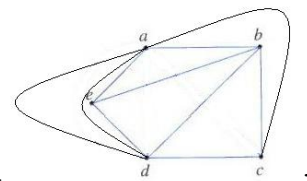**Eg. Show that the graphs are planar by rewriting them so no edges cross.**

(Sol.) (a)→ and (b)→
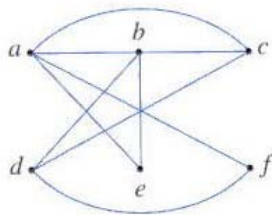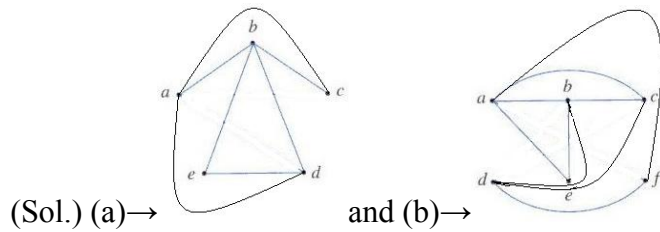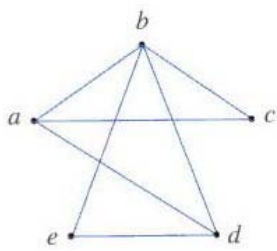
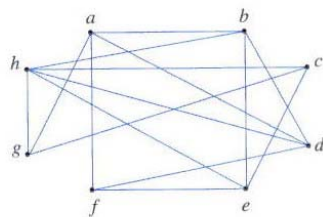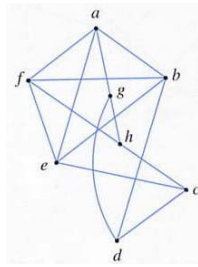**Eg. Determine whether each graph is planar?**

(Sol.)

(a)→ and (b)→ :
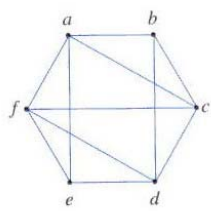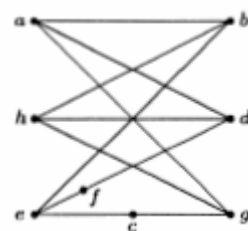
yes, but (c)→ : no

Homeomorphic graphs: Graphs G_1 and G_2 are homeomorphic if G_1 and G_2 can be reduced to isomorphic graphs by performing a sequence of series reductions.
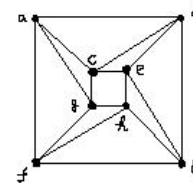

Eg. Show that the left graph contains a subgraph which is homeomorphic to $K_{3,3}$.

(Sol.)
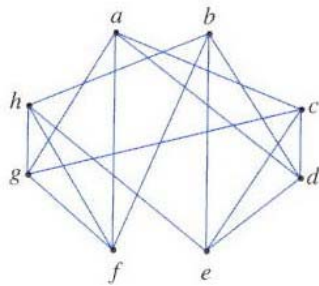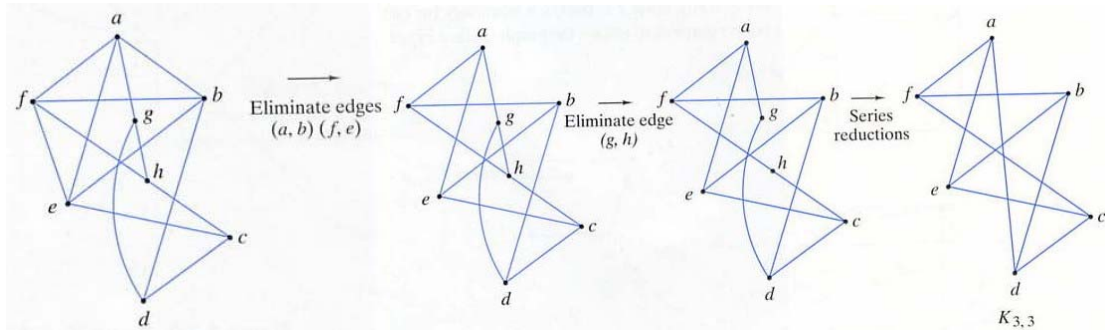
Eg. Determine whether the left graph is planar?
(Sol.) yes, it is planar and homeomorphic to the following graph.
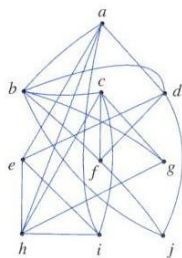
Eg. Determine whether the left graph is planar?
(Sol.) no

Euler's formula for graphs: $f=e-v+2$ **if G is a connected, planar graph with e edges, v vertices, and f faces.**

Eg. A connected, planar graph has 9 vertices with degrees 2, 2, 2, 3, 3, 3, 4, 4, and 5. How many edges are there? How many faces are there?

(Sol.) $\sum_{i=1}^{n} \delta(v_i) = 2+2+2+3+3+3+4+4+5 = 28 = 2e$, $e=14$, $f=e-v+2=14-9+2=7$ faces.
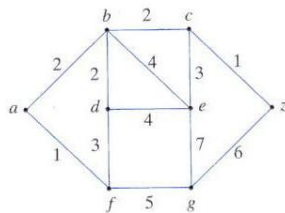
1-5 The Shortest Path

Dijkstra's shortest algorithm

Input: A connected, weighted graph in which all
        weights are positive; vertices a and z

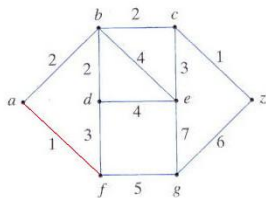Output: $L(z)$, the length of a shortest path from a to z

```
1.   dijkstra(w, a, z, L) {
2.       L(a) = 0
3.       for all vertices x ≠ a
4.           L(x) = ∞
5.       T = set of all vertices
6.       // T is the set of vertices whose shortest
7.       // distance from a has not been found
8.       while (z ∈ T) {
9.           choose v ∈ T with minimum L(v)
10.          T = T − {v}
11.          for each x ∈ T adjacent to v
12.              L(x) = min{L(x), L(v) + w(v, x)}
13.      }
14.  }
```

Eg. Find the shortest path from a to z.

(Sol.) $L(b)=L(c)=L(d)=L(e)$

$=L(f)=L(g)=L(z)=\infty$

Iteration 0:

$L(a)=0$,

$L(b)=\min[L(b),L(a)+2]$

$=\min[\infty,0+2]=2$,

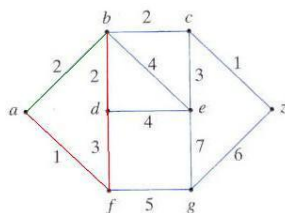$L(f)=\min[L(f),L(a)+1]=\min[\infty,0+1]=1$, $\therefore$ Select a,f
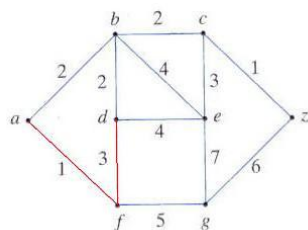
Iteration 1:

$L(d)=\min[L(d),L(b)+2,L(f)+3,L(e)+4]$

$=\min[\infty,2+2,1+3,\infty+4]=4$,

$L(g)=\min[L(g),L(f)+5,L(e)+7,L(z)+6]$

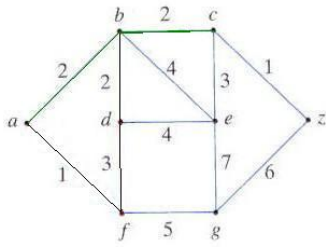$=\min[\infty,1+5,\infty+7,\infty+6]= 6$, $\therefore$ Select a,f,d

Iteration 2:

$L(b)=\min[L(b), L(d)+2, L(c)+2,L(e)+4]$

$=\min[2,4+2,\infty+2,\infty+4]=2$,

$L(e)=\min[L(e),L(d)+4,L(b)+4,L(g)+7]$

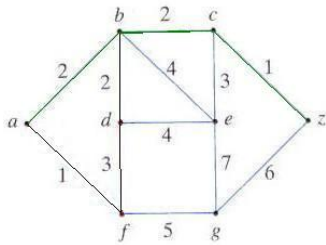$=\min[\infty,4+4,2+4,6+7]=6$, $\therefore$ Discard a,f,d and select $a,$

b

Iteration 3:

$L(c)$=min$[L(c),L(b)+2,L(e)+3,L(z)+1]$

=min$[\infty,2+2,6+3,\infty+1]$=4,

$L(e)$

=min$[L(e),L(b)+4,L(c)+3,L(d)+4,L(g)+7]$

=min$[6,2+4,4+3,4+4,6+7]$=6, ∴ Select a, b, c

Iteration 4:

$L(z)$=min$[L(z),L(c)+1,L(g)+6]$=min$[\infty,4+1,6+6]$=5,

$L(e)$=min$[L(e),L(c)+3]$=min$[6,4+3]$=6 ∴Select a, b, c, z, and the minimum length=5

The shortest path from a to z: $a{\rightarrow}b{\rightarrow}c{\rightarrow}z$ and the minimum length=5
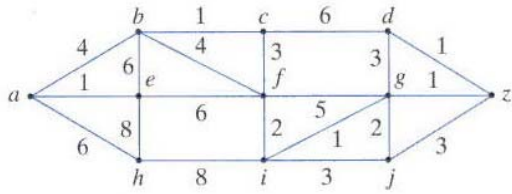
Eg. Find a shortest path from *a* to *z*.
(Sol.)
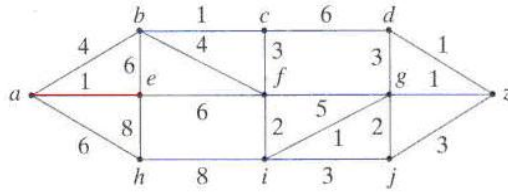
Iteration 0:

$L(a)=0$,

$L(b)=\min[L(b),L(a)+4]=\min[\infty,0+4]=4$,

$L(e)=\min[L(e),L(a)+1]=\min[\infty,0+1]=1$,

$L(h)=\min[L(h),L(a)+6]=\min[\infty,0+6]=6$,

$L(c)=L(d)=L(f)=L(g)=L(i)=L(j)=L(z)=\infty$, $\therefore$ Select *a, e*
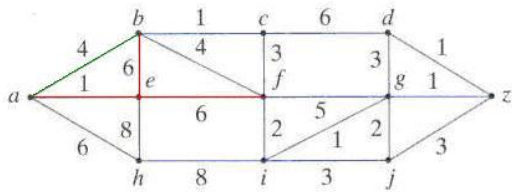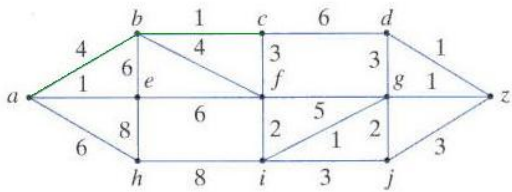
Iteration 1:

$L(f)$

$=\min[L(f),L(b)+4,L(e)+6,L(c)+3,L(i)+2]$

$=\min[\infty,4+4,1+6,\infty+3,\infty+2]=7$,

$L(b)=\min[L(b),L(e)+6]=\min[4,1+6]=4$,

$L(h)=\min[L(h),L(e)+8]=\min[6,1+8]=6$,

$\therefore$ Discard *a, e* and select *a, b*

Iteration 2:

$L(c)=\min[L(c),L(f)+3,L(b)+1,L(d)+6]$

$=\min[\infty,7+3,4+1,\infty+6]=5$,

$L(f)$

$=\min[L(f),L(b)+4,L(e)+6,L(c)+3,L(i)+2]$

$=\min[7,4+4,1+6,5+3,\infty+2]=7$
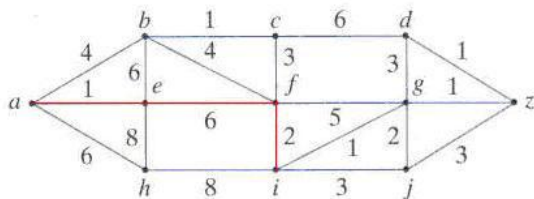

$\therefore$ Select *a, b, c*
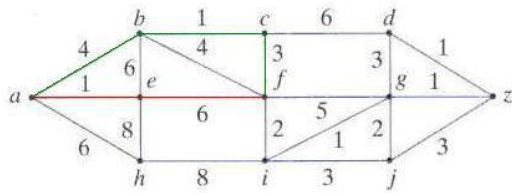
Iteration 3:

$L(f)$

$=\min[L(f),L(b)+4,L(e)+6,L(c)+3,L(i)+2]$

$=\min[7,4+4,1+6,7+3,\infty+2]=7$

$L(d)=\min[L(d),L(c)+6,L(g)+3]=\min[\infty,7+6,\infty+3]=13$, $\therefore$Discard *a, b, c, f* and select *a, e, f*
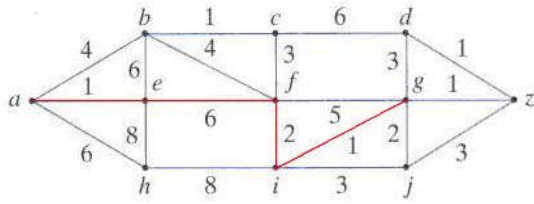
Iteration 4:

$L(i)=\min[L(i),L(f)+2,L(h)+8,L(j)+3]$
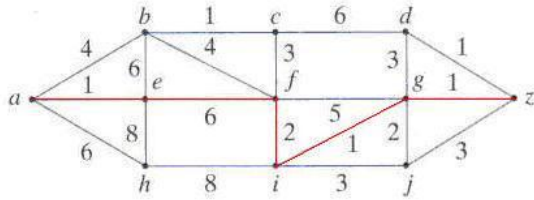
$=\min[\infty,7+2,6+8,\infty+3]=9$,

$L(g)=\min[L(g),L(f)+5,L(i)+1,L(j)+2]$

$=\min[\infty,7+5,9+1,\infty+2]=10$, $\therefore$ Select *a, e,*

f, i

e, f, i, g

Iteration 5:

$L(g)=\min[L(g),L(f)+5,L(i)+1,L(j)+2]$

$=\min[10,7+5,9+1,\infty+2]=10$,

$L(j)=\min[L(j),L(i)+3,L(g)+2,L(z)+3]$

$=\min[\infty,9+3,10+2,\infty+3]=12$, $\therefore$ Select a,

Iteration 6:

$L(z)=\min[L(z),L(g)+1,L(d)+1]$

$=\min[\infty,10+1,\infty+1]=11$,

$L(d)=\min[L(d),L(g)+1,L(c)+6]$

$=\min[\infty,10+3,7+6]=13$, $\therefore$ a, e, f, i, g, z

$\therefore$ The shortest path is $a{\rightarrow}e{\rightarrow}f{\rightarrow}i{\rightarrow}g{\rightarrow}z$ and the minimum length=11.

1-6 Graph Coloring and Chromatic Polynomials

Proper coloring: If $G(V,E)$ is an undirected graph, a proper coloring of G occurs when vertices a and b are colored with distinct colors if $\{a,b\}$ is an edge in G.

Chromatic number $\chi(G)$: The minimum number of colors needed to properly color G.

Eg. Find $\chi(G)$ for the left graphs. [台科大資工所]

(Sol.) (a) 2, (b) 3, (c) 4
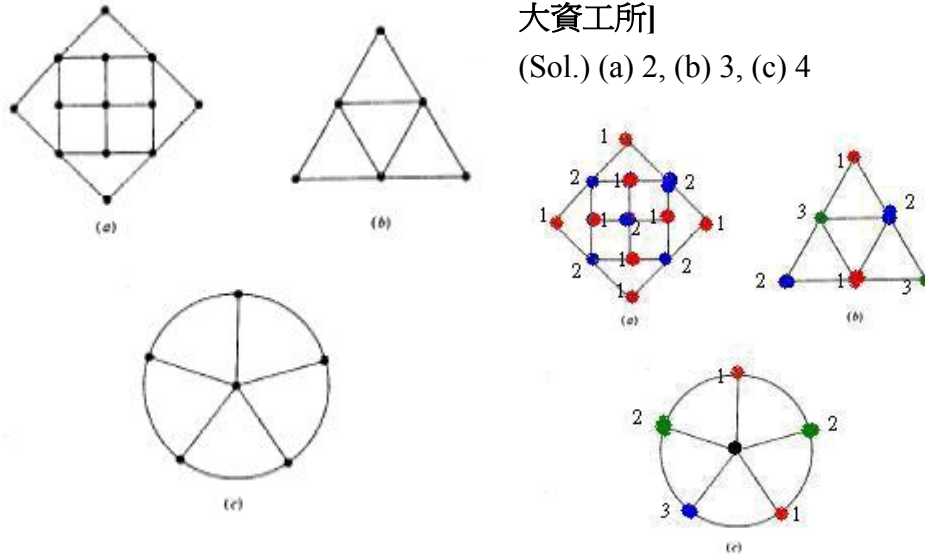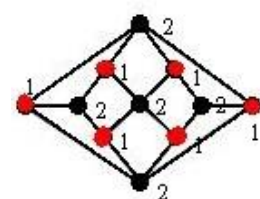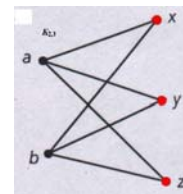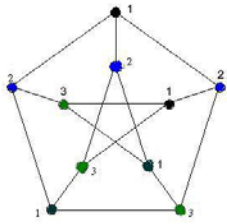

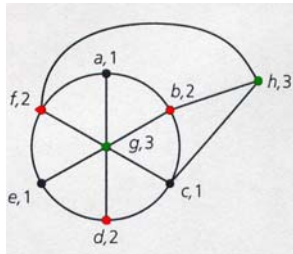**Eg. $\chi(K_n)=n$ and $\chi(K_{m,n})=2$.**

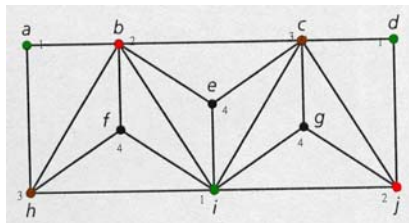**Eg. $\chi(G)=2$ for the Herschel graph.**

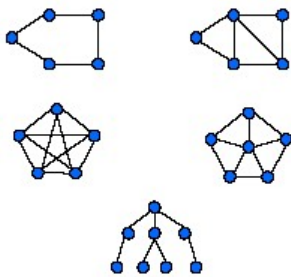**Eg. $\chi(G)=3$ for the Petersen graph.**

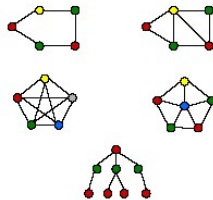**Eg. Find $\chi(G)$ for the left graph.**
(Sol.) $\chi(G)=3$

Eg. Find $\chi(G)$ for the left graph.
(Sol.) $\chi(G)=4$

Eg. Find $\chi(G)$ for the left graphs.
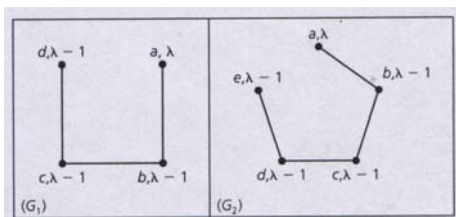(Sol.) (a)UL: 3, UR: 3, ML: 5, MR: 4, B:2

Chromatic polynomial $P(G,\lambda)$: Using at most λ colors, the number of ways of properly coloring the vertices of G. ($\lambda \geq \chi(G)$)

Theorem For G consists of n isolated points, $P(G,\lambda)= \lambda^n$.

Theorem For $G=K_n$, (a) $P(G,\lambda)=\lambda(\lambda-1)(\lambda-2)(\lambda-3)\ldots(\lambda-n+1)$ if $\lambda>n$. (b) $P(G,\lambda)=0$ if $\lambda<n$. (c) $P(G,\lambda)>0$ for the first time when $\lambda=n=\chi(G)$.

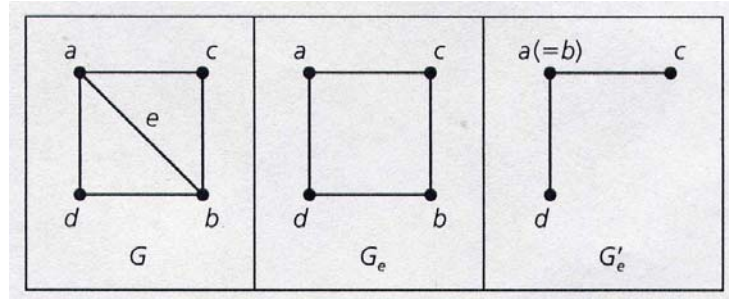**Theorem If G is a path on n vertices, $P(G,\lambda)=\lambda(\lambda-1)^{n-1}$.**

Eg. For the following graphs, $P(G_1,\lambda)=\lambda(\lambda-1)^3$ and $P(G_2,\lambda)=\lambda(\lambda-1)^4$.

Theorem If G is made up of components $G_1, ..., G_n$, $P(G,\lambda)=P(G_1,\lambda)...P(G_n,\lambda)$.

G, G_e, G_e': If $G(V,E)$ is a connected graph and $e=(a,b)\in E$, $G_e=G-e$ is a subgraph of G obtained by deleting e from G, G_e' is a subgraph of G obtained by coalescing the vertices a and b.

Eg. An example of G, G_e, and G_e'.

Decomposition Theorem for Chromatic Polynomials If $G(V,E)$ is a connected graph and $e\in E$, then $P(G,\lambda)=P(G_e,\lambda)-P(G_e',\lambda)$ or $P(G_e,\lambda)=P(G,\lambda)+P(G_e',\lambda)$, where $e=(a,b)$, $G_e=G-e$, G_e' is a subgraph of G obtained by coalescing the vertices a and b.

Eg. Obtain $P(G,\lambda)$ for G a cycle of length 4.

(Sol.)$P(G,\lambda)=P(G_e,\lambda)-P(G_e',\lambda)$
$=\lambda(\lambda-1)^{4-1}-\lambda(\lambda-1)(\lambda-2)$
$=\lambda^4-4\lambda^3+6\lambda^2-3\lambda$
$P(G,1)=0, P(G,2)=2>0\Rightarrow\chi(G)=2$.

Eg. Obtain $P(G,\lambda)$ for the following leftmost graph.

(Sol.) $P(G,\lambda)= P(G_e,\lambda)-P(G_e',\lambda)$
$=\lambda(\lambda-1)^{5-1}-(\lambda^4-4\lambda^3+6\lambda^2-3\lambda)$
$=\lambda(\lambda-1)(\lambda^3-4\lambda^2+6\lambda-4)$
$=\lambda(\lambda-1)(\lambda-2)(\lambda^2-2\lambda+2)$

Eg. Obtain $P(G,\lambda)$ for the following leftmost graph.

(Sol.) $P(G,\lambda)=\lambda\cdot\lambda(\lambda-1)(\lambda-2)(\lambda-3)-2\lambda(\lambda-1)(\lambda-2)(\lambda-3)=\lambda(\lambda-1)(\lambda-2)^2(\lambda-3)$
$P(G,1)=P(G,2)=P(G,3)=0$, but $P(G,4)>0$ for $\lambda\geq4\Rightarrow\chi(G)=4$.
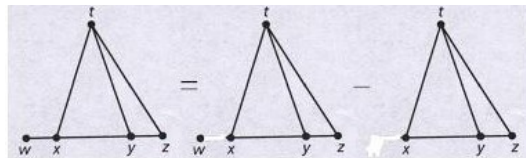
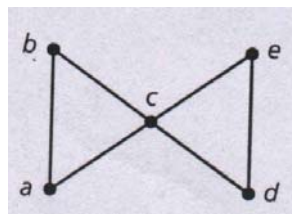**Eg. Obtain $P(G,\lambda)$ for the following leftmost graph.**
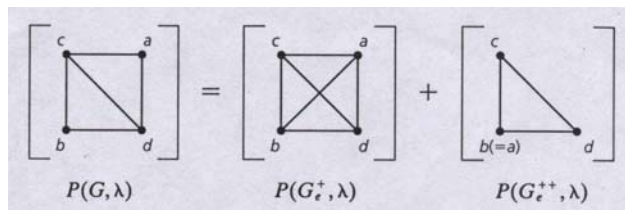
(Sol.) $P(G,\lambda)= \lambda(\lambda-1)(\lambda-2)(\lambda-3)$
$+\lambda(\lambda-1)(\lambda-2)=\lambda(\lambda-1)(\lambda-2)^2$
$\Rightarrow \chi(G)=3$.
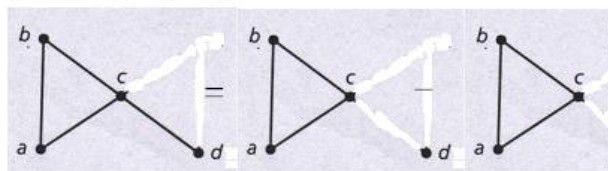If 6 colors are available, we have
$P(G,6)=480$ ways.

Eg. Obtain $P(G,\lambda)$ for the following leftmost graph.

(Sol.) $P(G,\lambda)=\lambda \cdot \lambda(\lambda-1)(\lambda-2)^2-\lambda(\lambda-1)(\lambda-2)^2$
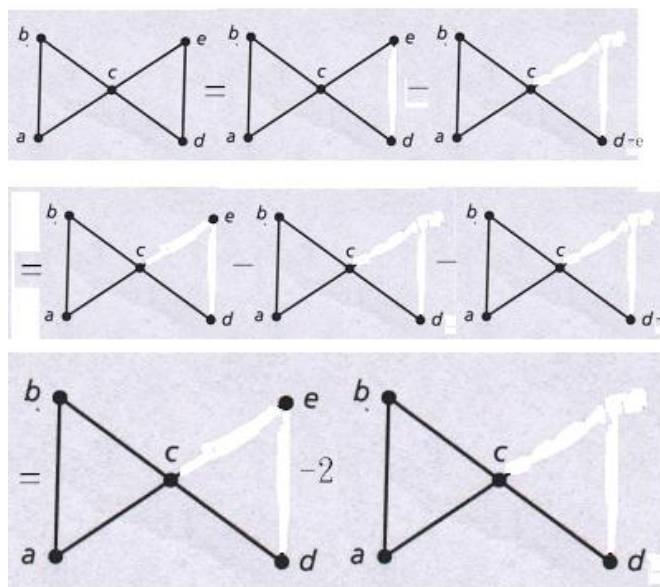$=\lambda(\lambda-1)^2(\lambda-2)^2$

Eg. Obtain $P(G,\lambda)$ for the left graph.
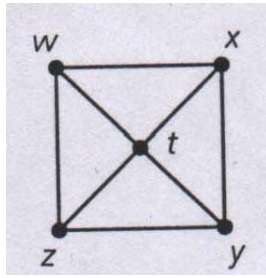

(Sol.) Firstly, we calculate
$P(G_1,\lambda)=\lambda \cdot \lambda(\lambda-1)(\lambda-2)- \lambda(\lambda-1)(\lambda-2)$
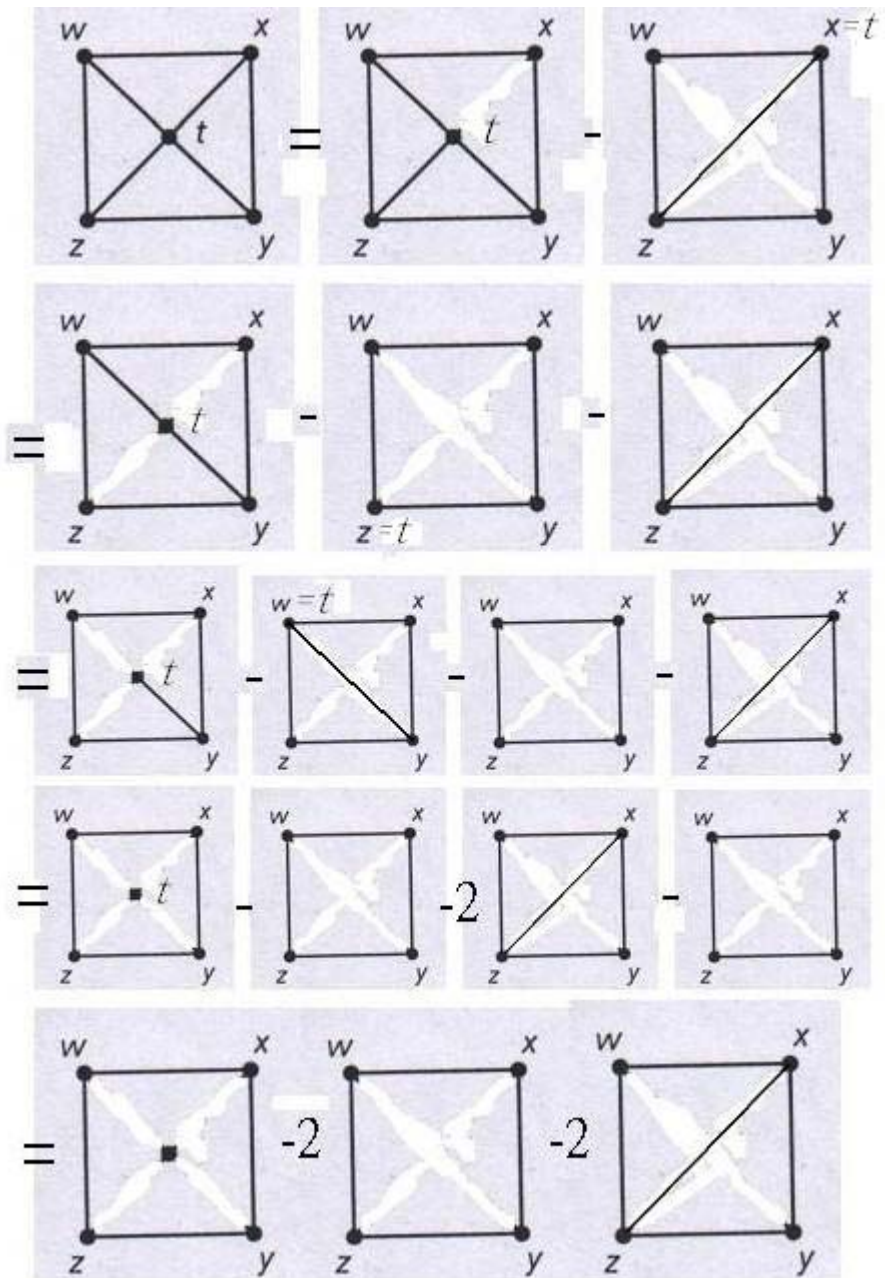$=\lambda(\lambda-1)^2(\lambda-2)$

And then we have

$P(G,\lambda)= \lambda \cdot P(G_1,\lambda)-2P(G_1,\lambda)$
$=(\lambda-2)P(G_1,\lambda)=\lambda(\lambda-1)^2(\lambda-2)^2$

**Eg. Obtain $P(G,\lambda)$ for the left graph.**

(Sol.) See the following figure.

$P(G,\lambda)= \lambda \cdot (\lambda^4-4\lambda^3+6\lambda^2-3\lambda)-2(\lambda^4-4\lambda^3+6\lambda^2-3\lambda)-2\lambda(\lambda-1)(\lambda-2)^2$

$=\lambda(\lambda-1)(\lambda^3-7\lambda^2+17\lambda-14)=\lambda(\lambda-1)(\lambda-2)(\lambda^2-5\lambda+7)$

Theorem Let $G(V,E)$ be an undirected graph with subgraphs G_1 and G_2. If $G_1 \cup G_2 = G$ and $G_1 \cap G_2 = K_n$, then $P(G,\lambda) = P(G_1,\lambda)P(G_2,\lambda)/\lambda(\lambda-1)(\lambda-2)(\lambda-3)\ldots(\lambda-n+1)$.
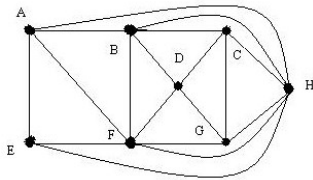
Dual graph: For the dual graph G' of the original graph G, its vertices consists of one point in each faces of G, including the unbounded face. An edge in G' connects two vertices if the corresponding faces in G have the common boundary.

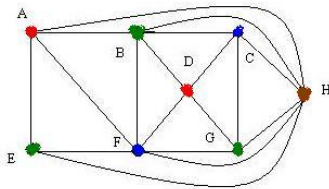**Application of coloring dual graph:** **Coloring the map**
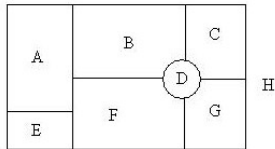
Eg. Find the dual graph of the left map and color the map using 4 colors such that no countries with adjoining boundaries have the same color.

(Sol.) 1. Transform the original map into the dual graph:

2. Color the dual graph: A, D: red, B, E, G: green, C, F: blue, H: brown

3. Color the original map: